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Shape optimization of one layer lattice shells using Simulated Anealling.

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1. Abstract
This article presents the procedure to obtain optimum designs in one layer lattice shells. The objective is to find parameters of form that have more effect on cost, of way comparable to the plane structures. from the exploration of specific solutions to the optimization of shells problem, using Simulated Annealing (SA).
These parameters are tried to determine according to the Theory of Design of Structures. The Theory tries that, in the first stages of the design process, the structural suitability can be insured from geometric parameters with guarantee of later validity. This Theory uses structural volume as objective function. Its definition and characteristics will be seen.
This article begin with the definition of shell, and a brief revision to SA (numerical approximation method that finds solutions near optimum), later the problem of optimization of one layer lattice shells, changing its geometry and topology, is exposed, some numerical examples are proposed and its solutions.
To finish diverse conclusions are extracted, as that SA is an appropriated method for this kind of problems, and the parameters of form, object of this investigation, are extracted from the found solutions.
2. Keywords: Structural optimization, Lattice shells, Simulated annealing.

3. Introduction
A shell structure is one that saves large span with small thickness, and this is made possible, fundamentally, due to its form. This is one of the most important factors of shells design. Forms are sought so that the cost diminishes, from a point of view of the quantity of used material, leaving aside aspects as manpower, the complexity of execution, or energetic aspects. The cost related to material consumption closely approaches the sustainable resources problem.
This is a complex problem with a high number of variables. We will try to reduce the problem by means of certain simplifications that will be exposed.

4. Simulated Annealing
To perform the search for solutions, a program that uses SA has been created. It is an stochastic approximation algorithm towards optimal solution. It does not use objective function derivates; it only evaluates the function to be optimised. SA receives its name for being an analogy with thermodynamic systems such as crystallization of a substance dissolved in water, or solidification of a metal (annealing). This process begins with a large temperature in which all the molecules of melted solid can move freely with respect to the others. When the melt is cooled slowly, this mobility is gradually lost. If the process has been sufficiently slow, particles are arranged tidily and approach a state of minimal energy. In some cases these states can be recognised by its visual perfection, such as the case of the form of pure crystals, in which its particles are perfectly in order. Annealing simulates a global improvement based on interactions with local changes that affect the nearby particles.
In case of optimization, temperature is an energy of reference, or parameter of control (ck in Figure. 1), that determines the fluctuation allowed in the studied configuration.
With this method, from an initial solution, an alternative is generated randomly, producing a change in one of the variables. To accept the initial configuration or its alternative, the Metropolis criterion (1) is applied. It is a probabilistic acceptance criterion: if the change is to the better, it is always accepted; but if the change is to the worse, it is only sometimes accepted. At the beginning of the process, virtually, all the transitions are accepted (high temperatures), whereas towards the end, changes to worse have a low probability of being accepted. The process ends when any change produces a null design probability of being accepted.

$$\begin{array}{lll} \textit{if} \;\; \boldsymbol{f}(\underline{x}_j) \le \boldsymbol{f}(\underline{x}_i) & \underline{x}_j & \textit{is accepted} \\ \textit{if} \;\; \boldsymbol{f}(\underline{x}_j) > \boldsymbol{f}(\underline{x}_i) & \underline{x}_j & \textit{is accepted with a probability of: } \exp((\boldsymbol{f}(\underline{x}_i) - \boldsymbol{f}(\underline{x}_j))/\, c_k) \end{array} \tag{1}$$

being:

$\underline{\boldsymbol{x}}_i$	actual configuration.
$\underline{\boldsymbol{x}}_j$	alternative configuration.
$\boldsymbol{f}(\underline{x}_i)$, $\boldsymbol{f}(\underline{x}_j)$	function's value in the states ***i*** and ***j*** respectively.
c_k	control parameter.

The feature of accepting worse solutions means that the simulated annealing can escape from local minimum. With initially high control values of the parameter, it is possible to guarantee that the search reaches configurations that are distant to the initial. This allows reaching areas of objective function with a better general behaviour. Besides, in many cases, when the obtaining of the optimum can be slow, SA obtains a reasonable approximation in a reasonable computational timeframe.
Kirkpatrick[6] used SA for the first time in the optimization of integrated circuits. Nowadays it is used in many fields of study, in very difficult combinatorial problems with good results. To find the demonstration of the convergence[7] and the programming of algorithm[8] see the bibliography.

```
procedure SIMULATED _ANNEALING;
begin
  INICIALIZE (x0, c0, M 0);
  C:=0;
  xi:= x0;
  repeat
    for v:=1 to M  do
    begin
      GENERATE_ALTERNATIVE (xj from xi);
      if f (xj) ≤ f(xi)  then xi:= xj
      else
      if exp((f(xi)- f(xj))/ ck)>random[0.1) then xi:= xj
    end;
    C:=C+1
    CALCULATE_TRANSITIONS (M);
    CALCULATE_CONTROL (ck);
  until stop_criterion
end;
```

Figure. 1: A pseudo language program of the SA[5]

5. Objective function.

The objective function used is the Structural Volume (*Vs*), and it is defined as the integral extended to the whole structure, of the product of the absolute value of the axial strain in the section, ***N*,** by **ds**, the longitudinal element of the piece in the considered point. It simultaneously measures forces and its trajectory, its dimensions are those of work. Its expression in a constant and discrete way is shown here.

$$Vs = \int |N| \cdot ds \qquad Vs = \sum_{i=1}^{e} |N_i| L_i \tag{2}$$

The importance of this function is that it has good relations with other magnitudes which are used frequently to evaluate the cost of a structure: its volume or weight. Simultaneously *Vs* is abstracted of the own properties of the material and it only considers aspects of geometry, topology and applied loads. For this reason, *Vs* contributes form considerations besides that of the material.

In the field of structural optimisation conventionally is used the Material Volume (*Vm*), its definition is equivalent under assumption of use only one material in a structure and

$$V_m = \sum_{i=1}^{e} A_i \cdot L_i = \sum_{i=1}^{e} \frac{|N_i|}{f} \cdot L_i \; ; \tag{3}$$

Where:

f: material strain (symmetrical to traction and compression or an average value)

Ai: Cross-section area of every *i* element.

In the case of a structure with bar elements and full-stressed design:

$$V_m = \frac{Vs}{f} \tag{4}$$

6. Optimization of shells: Bases.

The aim is to find the configuration of geometry and topology that makes the structural volume of shell as small as possible. The object of design consists of a lattice shell with a single layer, like a surface triangled by bars. Its form is defined by its geometry, **g**, and topology, **t**.

The geometry is determined by $\mathbf{g}(n)=\{(x_1,y_1,z_1),...,(x_n,y_n,z_n)\}$, which represents the coordinates in the space of *n* nodes and varies inside **G**, all possible coordinates of *n* nodes in a previously defined region, ($\mathbf{g} \in \mathbf{G}$). Total number of nodes, *n*, remains constant throughout the whole process.

Structure topology defines the way in which the nodes are connected to others, it is possible to express it as the set of bars that join two nodes $\mathbf{t}(e) = \{(n_{1i}, n_{1j}),...,(n_{ei}, n_{ej})\}$, *e* is the number of bars being a fixed value throughout the process. Topology varies within all the possible topologies ($\mathbf{t} \in \mathbf{T}$)

Only lattice shells that are internally statically determined are taken into consideration. Besides, the sustentation cannot be redundat, except those cases in which symmetrical conditions are guaranteed. This entails differences in how forces are materialized in the foundations (for example in traction rings) that can conditions that are not comparable among the forms in competition.

The set of constraints is defined of the following way: $\mathbf{v} = \{(n_1, \delta_x), (n_1, \delta_y), (n_1, \delta_z),..., (n_f, \delta_x), (n_f, \delta_y), (n_f, \delta_z)\}$, each couple consists of the node number in which the constraint is applied and the direction of the constraint. In this set all the fixed nodes appear at least once (n_f being the number of fixed nodes ($n \geq n_f$)).

The supported load is defined by surface unit and is applied in vertical direction, Z axis. Load is distributed to all nodes in a joint

load, $\mathbf{q}(i)$.
For the analysis, the finite elements method is applied with bar type elements of three freedom degrees by node (displacements in all three coordinated axes).
From the analysis the cross-section areas are sized. When the structure is statically determined, the area depends on the axial strain of each bar and the safe tension of the material, *f*. When the structure is not statically determined, during the analysis areas have been considered as equal for all bars, the definitive transversal cross-section area is obtained as in the previous case.

6.1. General formulation of the design problem.
The degrees of freedom of a configuration can be expressed as $\mathbf{x}$, with $\mathbf{x}=\{\mathbf{g}, \mathbf{t}\}\in\{\mathbf{G}, \mathbf{T}\}$, where $\mathbf{G}$ and $\mathbf{T}$ are sets of all possible geometries and topologies respectively. Every set $\mathbf{x}$, corresponds with a solution, that in order to be valid must be contained inside the search domain, $\mathbf{X}$
In consequence, optimization problem can be expressed according to formula (6). That is, to find the configuration of geometry and topology that make the objective function minimal, within the search domain.
Find $\mathbf{x}_0$, such that:

$$f(\mathbf{x}_0) = \min_{\mathbf{x}\in\mathbf{X}}\left(f(\mathbf{x})\right) \tag{5}$$

6.2 Search domain.
The space of search, $\mathbf{X}$, is constituted by all the lattice shells with *n* number of nodes, the determined constrains, and fulfils the following requirements:
The structural requirements are three basic ones: resistance, stiffness and general stability. Nevertheless, the condition of local stability (buckling) has not been considered, making the solutions valid only from a theoretical point of view.
The geometric and topologic requirements are the following:

1. The interior nodes can change their position to any point inside the enclosure defined by two points that mark the lower and upper limits { (x_{min}, y_{min}, z_{min}), (x_{max}, y_{max}, z_{max})}.
2. All contour nodes have fixed coordinates.
3. Maximum bar length is limited to avoid that the problem of flexion be ignored, moving loaded nodes near to supports.
4. The bars always intersect in nodes and they never have coincidental projections, thus avoiding double layered solutions.

6.3 Generation of shells
With the alternatives generator and with a finite number of plays, it attempts to assure that there is not a null probability that a shell could be transformed into any other one, and that it continues fulfilling the conditions of the problem. Nevertheless, every transformation produces a small variation.
A specific mechanism of generation has been designed for each category of variables (Figure. 2): the first one being the geometry (1-change of node position) and the second one the topology (2-change a bar for another one). These two plays have a similar number of freedom degrees and for this reason both have the same probability of being chosen.

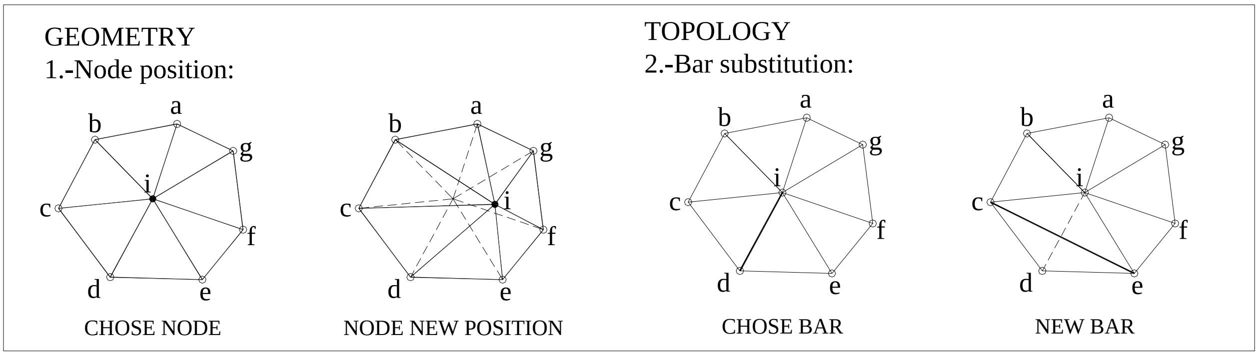

Figure. 2: Plays of the generator of alternatives.

7. Numerical Examples
A series of examples that solve simple problems are used, with a basic geometry: circular or square base, and a mixed geometry by addition of semicircular and square bases or by forming a rectangle. Load is always the same $4.5kN/m^2$. The studied case has a enclosures with levels $z\geq0$ (being $z=0$ the level of the supports, this way the solutions of hung membrane working fundamentally to traction are eliminated). The maximum length of bar is 40% of short span.
After the analysis process, the program gives solution in entry files to the ANSYS-ED 5.3 program (for the analysis of structures that uses the finite elements method). This way, the analysis results are confirmed and the shown images can be obtained, where it is possible to observe the way of working of the structure. In parallel, an AutoCAD image appears with lights and shades. Every set is labelled with structural volume in kN.m, which does not depend on the material used, but only on the axial strain and the bars lengths.
The initial form of the structure is not important if at the beginning of the process, temperature takes high values. Thus, big changes are produced at the beginning, which are normally accepted, although initial solutions are also showed.

7.1. Squared and rectangular base, initial configurations.

Three initial configurations are used: squared base(*Figure. 3* left), rectangular base with a proportion of 2/1 (*Figure. 3* centre), and rectangular base with a proportion of 3/1 (*Figure. 3* right). The first one covers a base of 100x100m, has 25 nodes and 60 elements, with 12 vertical sustentation constrains in contour nodes, and 4 horizontal sustentation constrains, symmetrically displayed in corners. This makes the set statically undetermined for sustentation with redundancy equal to 1.

The second one covers a base of 50 x 100m, has 46 nodes, 118 elements and 18 vertical sustentation constrains in contour nodes, and 4 horizontal sustentation constrains, symmetrically displayed in corners. The structure is statically undetermined for sustentation with redundancy equal to 2.

The third one covers a rectangular space of 50 x 150m., with 67 nodes and 175 elements. The 24 contour nodes have a constraint to vertical movement. Three horizontal constrains are added in the corners to prevent the displacement and the turn of the set as a rigid solid. Structure is statically undetermined for sustentation with redundancy equal to 2.

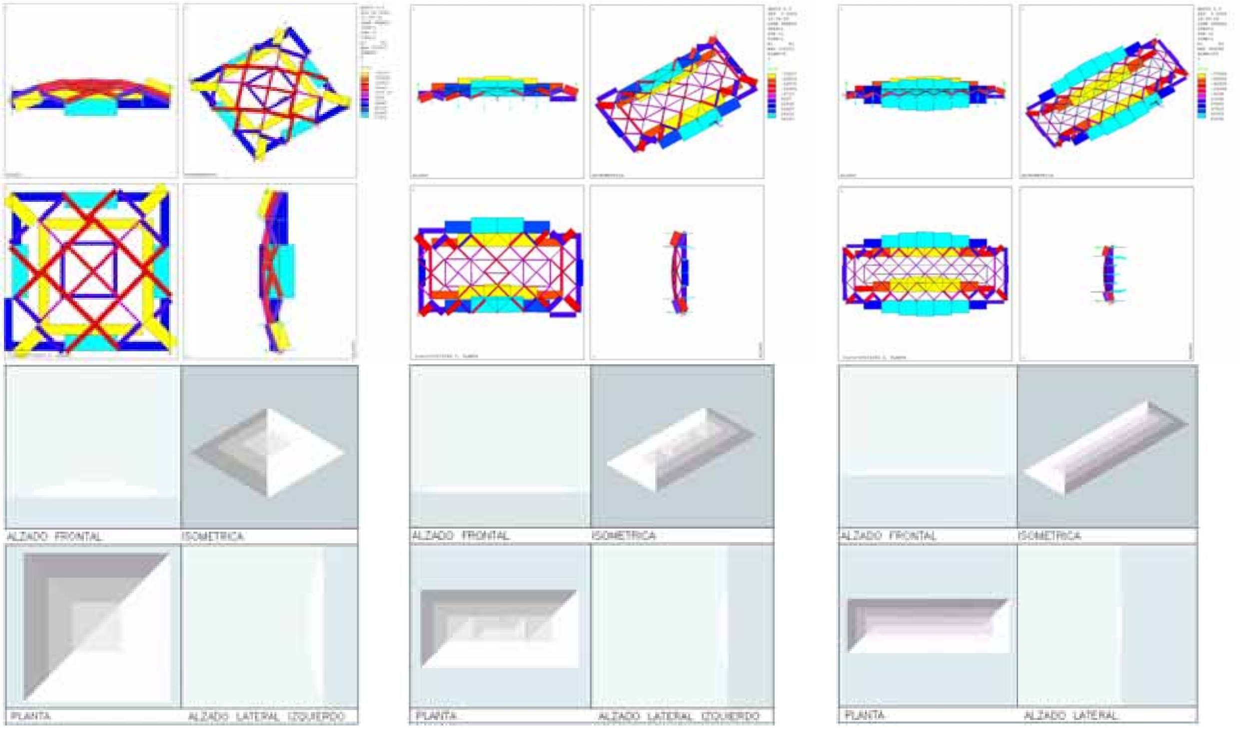

Figure. 3: Initial configuration: squared base (Vs=29.776 e06kN.m), rectangular base 2/1 (Vs=15.572 e06kN.m) and : rectangular base 3/1 (Vs=43.534 e06kN.m)

7.2. Squared and rectangular base solution.

When the base is squared, the solution has a dome form (Figure. 4 left) with bars working fundamentally on compression, without intermediate rings and a ring in the base on traction. To adapt the dome form to a square base, flat zones appear in corners. It looks like radial arches and surface folds that help in resisting to local flexions. The dome has a parabolic symmetrical figure.

When the base is rectangular 2/1 (Figure. 4 centre), the solution consists of two juxtaposed domes formed by bars working on compression and two rings in the base on traction. The major solicitations appear in the intersection of the rings. In the corners, the form tries to adapt to the rectangle by means of flat zones between the rings and the corners. The squared base solution behaviour is repeated.

When the base is rectangular 3/1, the solution has a series of three juxtaposed domes. Not all the domes have the same development, in the section, (Figure. 4 right), can observe that they all have, approximately, the same total rise and the right dome has less extension. To the right, the dome is solved by an arch towards the corner, since the ring cannot be closed.

In Figure. 4 the way of working of each element can be appreciated. Domes bars work fundamentally to compression and their respective rings at the base work to traction. To adapt the structure to a rectangular form, flat zones appear in corners. There is an additional intention of forming a ring with a circumference shape and, due to the limitation of bar length, this ring is a broken line.

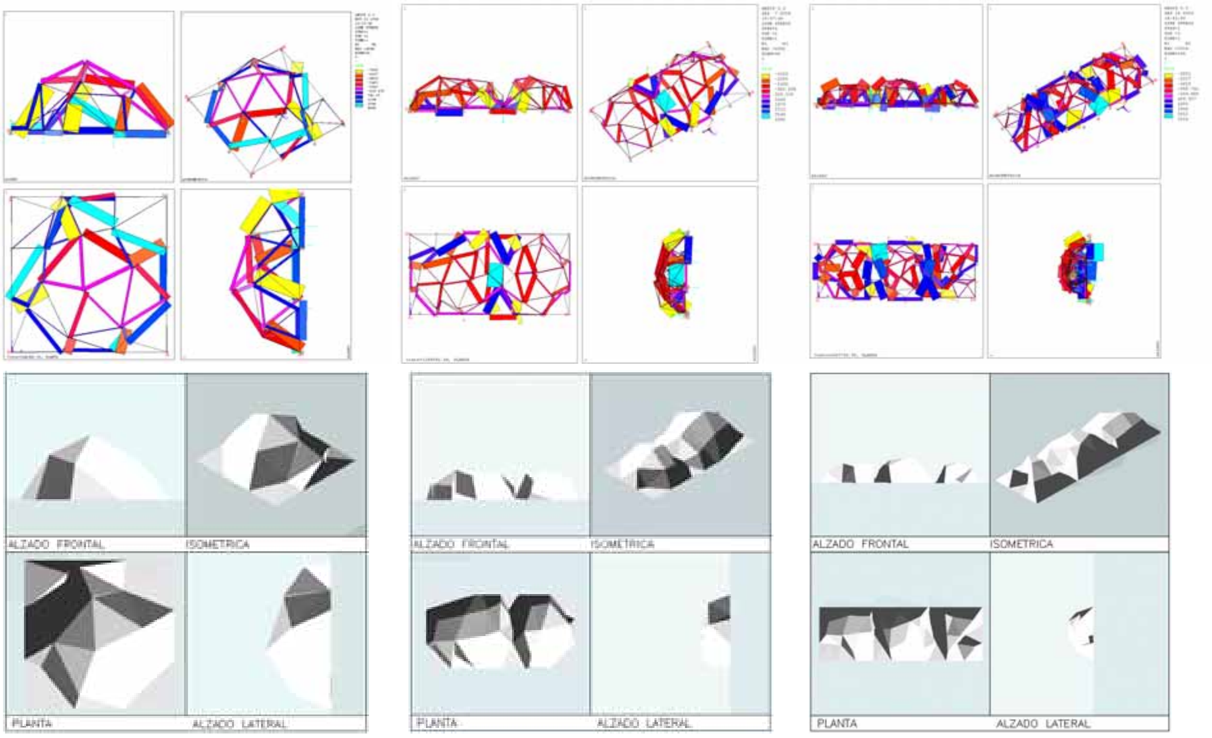

Figure. 4: Squared base solution, (Vs 3.442 e06 kN.m), 2/1 rectangular base solution, (Vs = 1.188 e06kN.m), 3/1 rectangular base solution, (Vs=1.868 e06kN.m.)

7.3 Circular and mixed base: initial configurations.

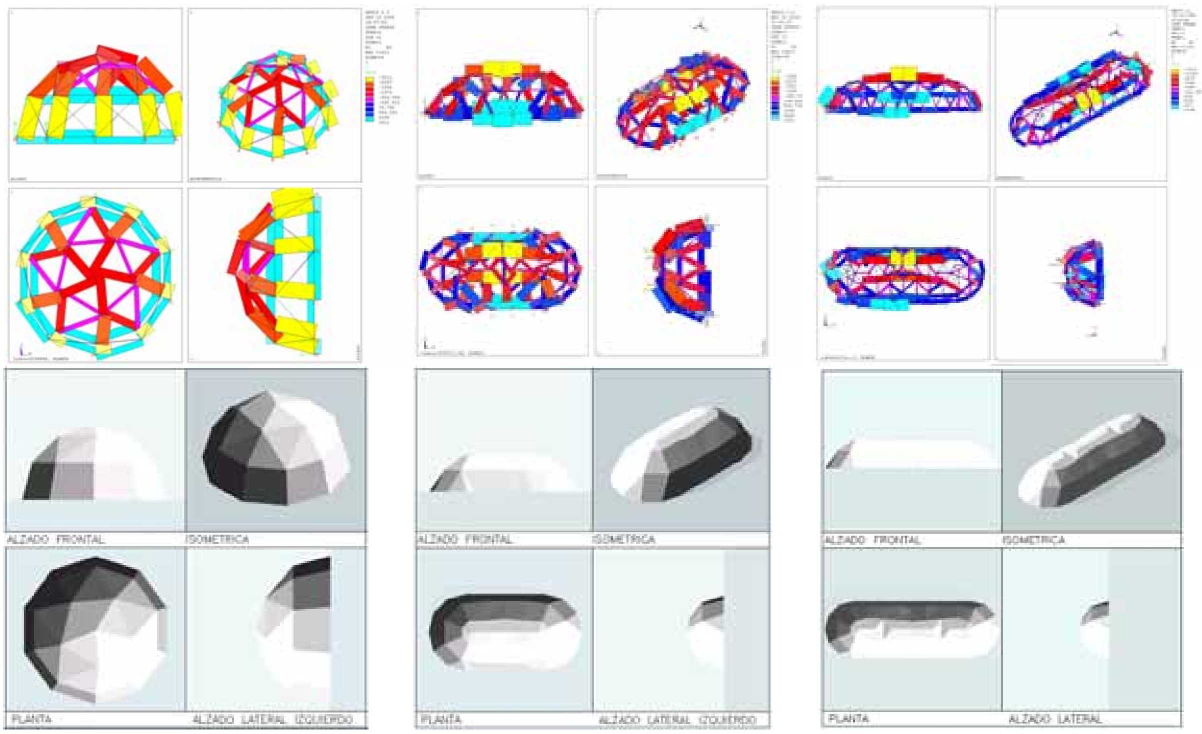

Figure. 5: Inicial configuración: circular base (Vs=2.599 e06kN.m), mixed base 2/1 (Vs=1.994 e06kN.m), mixed base 3/1 (Vs=9.257 e06kN.m).

Three initial configurations are used. The first one has a circular base (Figure. 5 left), and two more have a mixed base that combines semicircular ends with straight edges. They cover spaces of proportion 2/1 and 3/1 (Figure. 5 centre and right). The first configuration covers a base with a diameter of 100m, has 26 nodes, 65 elements and 13 sustentation constrains, which are distributed as follows: 10 in vertical in all contour nodes, and 3 horizontal tangent to circumference. Thus, the horizontal constrains do not take

part in the distribution of work. It is an statically determined structure.
The second configuration covers a space of 50x100m, has 54 nodes, 137 elements and 25 sustentation constrains of which 22 are vertical and 3 are horizontal tangent to base circumference. Again it is an statically determined structure.
The third configuration covers a space of 50x150m, has 82 nodes, 213 elements and 33 sustentation constrains; the structure is statically determined.

7.4 Circular base and mixed base solution.
With circular base the dome is parabolic and practically symmetrical. (Figure. 6 left) with bars working fundamentally on compression, without intermediate rings and a ring in the base on traction. It looks like radial arches and surface folds that help in resisting to local flexions.
The 2/1 mixed base solution is similar to the case of the rectangular base: two juxtaposed domes or a saw tooth cover. Two semi-rings appear in base, coinciding with the shape of the base, whereas in the straight edge, the low part of teeth, the strongest tractions appear (Figure. 6 centre).
The 3/1 mixed base solution has three juxtaposed domes of similar development. Again it uses the base form, forming two semi-rings in the ends. In the central zone the tractions are solved in a similar way to the previous example by means of a few bars on traction in the base of domes (Figure. 6 right)

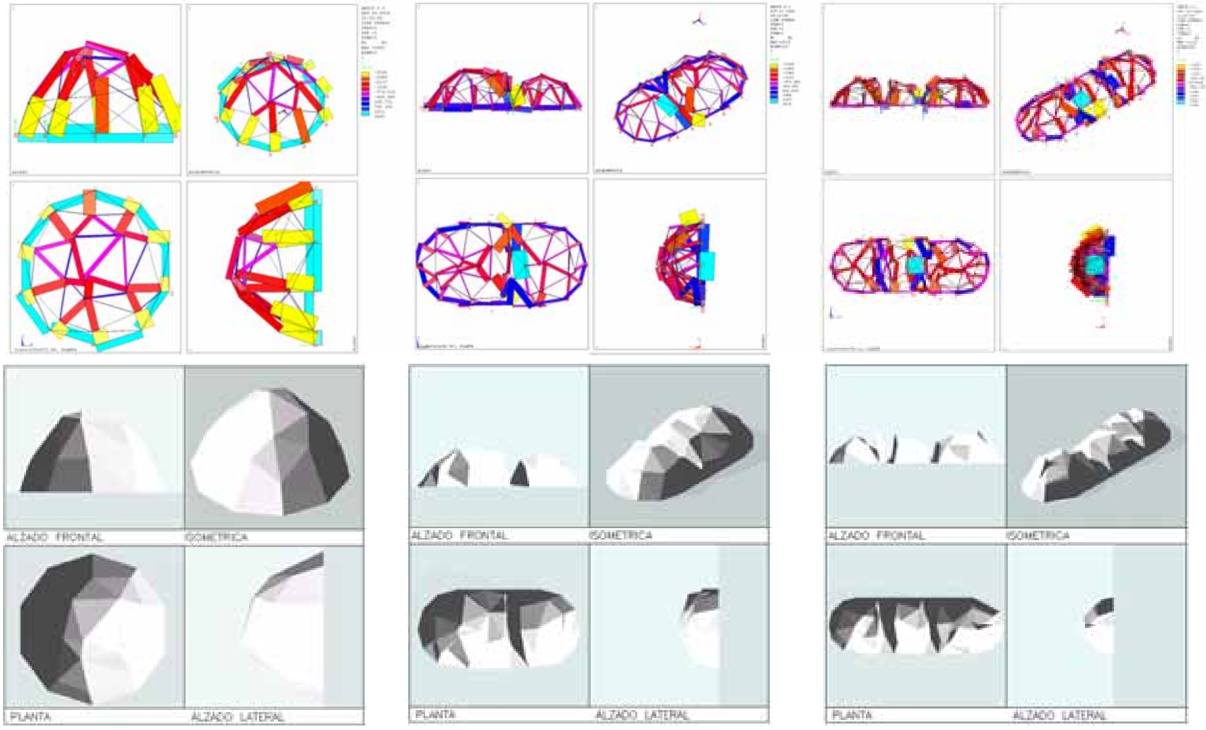

Figure. 6: Circular base solution (Vs=2.056 e06kN.m), 2/1mixed base solution (Vs= 0.924 e06 kN.m), 3/1mixed base solution (Vs= 1.442 e06 kN.m).

8. SUMMARY OF RESULTS

The obtained results are compared in the following table. It is possible to see that the structural volume diminished considerably in all the solutions, with regard to initial one.

	$Vs_{initial}$ /1e6 [kN.m]	Vs/1e6 [kN.m]	q*A*l [kN.m]	w	λ=l/h
SQUARED	29.776	3.442	4.5*10000*100	0.765	2.30
RECTANGULAR 2/1	15.572	1.188	4.5*5000*50	1.056	2.51
RECTANGULAR 3/1	43.534	1.868	4.5*7500*50	1.107	2.15
CIRCULAR	2.599	2.056	4.5*7853*100	0.582	1.88
MIXED 2/1	1.994	0.924	4.5*4463*50	0.920	2.05
MIXED 3/1	9.257	1.442	4.5*6963*50	0.921	1.92

Table 1: Summary of results

Values of structural volume (*Vs*) and slenderness *(l/h)* are indicated, where *h* is the total rise. To compare the values of structural volume, a non-dimensional magnitude named *reduced structural volume*, **w**, (7) is created by dividing the structural volume by the load per surface unit, *q*, the covered area, *A*, and the short saved span, *l*.

$$w = Vs / q * A * l \qquad (6)$$

9. CONCLUSIONS AND PARAMETERS

Observing the behaviour of SA in this and other[4] examples, it is possible to affirm that it allows the study of the search for optimum shells, and that in such a complex problem in which it is difficult to obtain the ideal structure, the algorithm behaves essentially well, contributing acceptable solutions and close to the optimum.

The optimization is carried on only for a uniform surface load, thus optimization for self-weight loads is not made. The results of the optimization process are very irregular in shape. The first example is the optimization of a symmetric square-base dome shell. The result is a fully unsymmetrical shape which is in strong contrast with the symmetry of the load and of the boundary conditions. The other examples show similar behaviour.

The studied solutions represent the continuity of forms that exists behind each one. This set is the one which is tried to be explored and consists in all the best solutions. From simplifications as the strict sizing of cross-section areas and the non-consideration of bucking, one can capture the properties that are of interest.

9.1. Solutions with dome form.

The solutions with the limitation of enclosures z≥0 are parabolic domes, internal push forces (thrust) are solved in the base ring. Due to this form intermediated rings are not needed. Considering the global flexion of collapse of the form, when comparing the initial structure (semi-sphere) with the solutions (paraboloid), one can observe (*Figure. 7*) the advantage of parabolic domes over semi-spheres; with equal rise, the global lever arm increases. For this reason, in the parabolic solution the lever arm has increased with respect to the initial one, and the work is reduced.

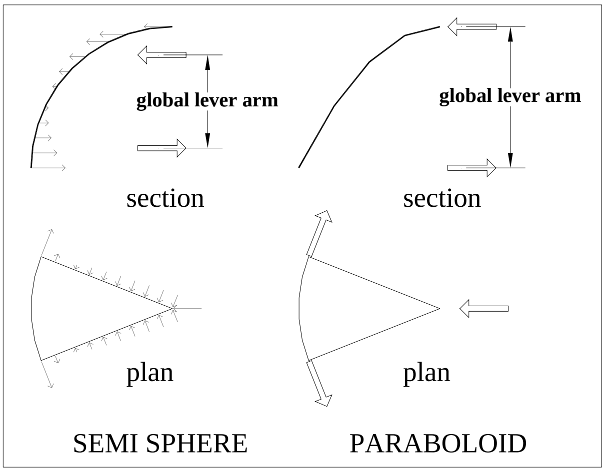

Figure. 7: Comparison to collapse of a structure sector .

9.2. Tension concentration.

In all solutions one can observe the concentration of tensions in a few elements that hierarchically arrange the solution into a series of principal elements: radial arches and a base ring. This derives from the discrete size of elements. On one hand it contrasts with the usual shell behaviour with continued tension distribution. On the other hand, the tension concentration is also a natural phenomenon in many occasions: a clear example appears in the leaves of plants.

9.3 Repetition rhythm

In the case of elongated proportions, one can observe a repetition of the simplest solutions, as often as the proportion of the rectangle’s sides. Generally speaking, one could say that the number of repetitions will depend on the proportion between the sides of the surface to be covered.

$$r = L/l \qquad (7)$$

L,l large and sort span respectively.
r number of solution repetitions

9.4 Cost coefficients depending on proportion

The values of ω (Table 1) have a relatively logical sequence: the more rectangular the shape, the costlier it is. One could say that in solutions of rectangular proportion there is an extra cost for having to solve the problem in only one direction—in central zones—whereas in extreme zones, as happens in solutions of squared proportion, the problem is solved in two directions.

Two cost coefficients can be established depending on the proportion of reduced structural volume destined to solve the problem in an unidirectional or bidirectional manner, which has relation with supporting itself on the whole contour or supporting itself on two opposite sides.

Cr work coefficient in one direction, or of supporting on opposite sides.

Cc work coefficient in two directions, or of supporting on the whole contour.

Figure. 8: Two ways of taking the load to the supports, in one or two directions.

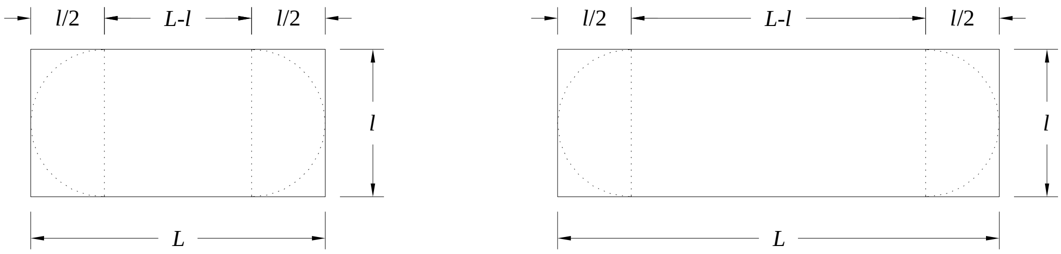

Figure. 9: Dimensions.

If square base solution destines 100 % to solve it in a bidirectional way, this coefficient would be Cc = 0.582

In a rectangular base of sides L and l of proportions 2/1 and 3/1, as in presented examples, the structural volume is distributed according to the following expression

$$\frac{(L-l)\cdot Cr + l\cdot Cc}{L} = \omega\,. \qquad (8)$$

For square and rectangular base: if Cc = 0.765 the value obtained for the coefficient of unidirectional work (Cr) is 1.35 in case of proportion 2/1 (10), and 1.29 in case of proportion 3/1 (11).

$$\frac{(L-l)\cdot Cr + l\cdot Cc}{L} = 0.5\cdot Cr + 0.5\cdot Cc = 1.056\,; \qquad Cr=1.35 \qquad (9)$$

$$\frac{(L-l)\cdot Cr + l\cdot Cc}{L} = 0.66\cdot Cr + 0.33\cdot Cc = 1.107\,; \qquad Cr=1.29 \qquad (10)$$

For circular and mixed base: if Cc = 0.582 the value obtained for the coefficient of unidirectional work (Cr) is 1.26 in case of proportion 2/1 (12), and 1.10 in case of proportion 3/1 (13).

$$\frac{(L-l)\cdot Cr + l\cdot Cc}{L} = 0.5\cdot Cr + 0.5\cdot Cc = 0.920\,; \qquad Cr=1.26 \qquad (11)$$

$$\frac{(L-l)\cdot Cr + l\cdot Cc}{L} = 0.66\cdot Cr + 0.33\cdot Cc = 0.921\,; \qquad Cr=1.10 \qquad (12)$$

Thus, costs that cannot be reduced in solutions of positive enclosure would be 0.6*q*A*l for the regions with support in perimeter, and between 1.1 and 1.35 times q*A*l for the regions with parallel supports. In order to do so, the solutions will try to form "bubbles" or domes of approximately similar forms in two orthogonal directions, materializing girders contained in its surface, which allow to save the span by forming the required folds.

9.5. Final conclusion.

The found parameters can be useful as a direct application towards new shell designs, in both lattice ones with linear elements (preferably for steel or wood) or superficial elements (fundamentally of reinforced concrete).

Because of the irregularity of the solutions, further work on the results and the causes of this irregularity is needed.

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